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Summary - A method is presented for the determination of the two-dimensional Green's functions for the scalar wave equation for a region between two parallel lines. These functions are useful for the consideration of electromagnetic parallel-wall waveguides. The approach is based on using electromagnetic boundary conditions. Author

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Introduction

In considering the wave propagation in waveguides, Green's functions are very useful for the determination of the distribution of the field components resulting from excitation by arbitrary sources. The Green's functions represent the field distributions resulting from excitation by point sources. A special case of a waveguide is the parallel-wall guide. Morse and Feshbach have shown the derivation of the Green's function for this type of waveguide by the method of electromagnetic images*. In the following considerations, a different method using electromagnetic boundary conditions is presented.

Original Conditions

A waveguide consisting of a pair of perfectly conducting walls is assumed with the walls parallel to the xz-plane and extending to infinity. The walls are located at $y = 0$ and $y = d$. The waves propagate between the walls in the direction of the z-axis. The corresponding geometrical conditions are indicated in Fig. 1. The waves propagating along such a structure can be considered generally as a superposition of orthogonal transverse electric (TE) and transverse magnetic (TM) wave modes. The components of the field intensities for these modes can be derived from the longitudinal components. In the absence of sources, these components satisfy the following wave equations:

$$\text{TE } (E_z \equiv 0): \quad \nabla^2 H_z + \eta^2 H_z = 0,$$

$$\text{TM } (H_z \equiv 0): \quad \nabla^2 E_z + \eta^2 E_z = 0, \qquad (1)$$

* P. M. Morse and H. Feshbach: Methods of Theoretical Physics. McGraw-Hill, New York, N. Y., 1953, pp. 812-818.

where $\eta^2 = k_o^2 - k_z^2$. The quantities k_o and k_z are the free-space and the longitudinal propagation constants ($k_o = 2\pi/\lambda_o$, $k_z = 2\pi/\lambda_g$). The boundary conditions corresponding to the geometrical configuration indicated in the figure are

$$y = 0,\ y = d: \qquad E_z = 0; \quad \frac{dH_z}{dy} = 0. \tag{2}$$

The general transverse distributions $H_z(x,y)$ and $E_z(x,y)$ which satisfy the wave equations [Eqs. (1)] and the boundary conditions [Eqs. (2)] have the form

$$H_z = \sum_{n=0}^{\infty} a_n \cos k_n y \, e^{\pm j k_x x}, \tag{3a}$$

and

$$E_z = \sum_{n=0}^{\infty} a_n' \sin k_n y \, e^{\pm j k_x x}, \tag{3b}$$

where $k_n = n\pi/d$. In the presence of sources in the yz-plane, the amplitudes a_n and a_n' depend on the distribution of these sources which are assumed to have infinite extension and to be uniform in the z-direction.

Method of Derivation

The approach followed in deriving the Green's functions is based on the following concept. First, the field intensities are determined for the Regions I and II. These regions are located above and below the yz-plane in which an infinite line current is assumed at the position y_o as a source. The line current is magnetic in the case of TE and electric for TM waves. The boundary conditions for the various field components at $x = 0$ yield then relationships between the amplitudes of the wave elements [Eqs. (3)] in Region I and II and relationships for their amplitudes depending on the structure of the source.

TE Waves Excited by a Line Source

The scalar Green's function is the solution of the wave equation in the presence of a point source at x_o, y_o. It satisfies the nonhomogeneous wave equation

$$\nabla^2 G + \eta^2 G = - \delta(x - x_o)\, \delta(y - y_o). \tag{4}$$

The corresponding wave equation for H_z is

$$\nabla_{tr}^2 H_z + \eta^2 H_z = j \frac{\eta^2}{\omega\mu} I_{zm}\, \delta(x - x_o)\, \delta(y - y_o), \tag{5}$$

where the source is represented by an infinite magnetic-line current I_{zm}. Comparison of Eqs. (4) and (5) indicates that after normalization, $I_{zm} = 1$, G and H_z are interrelated by

$$G^{TE} = + j \frac{\omega\mu}{\eta^2} H_z . \tag{6}$$

With the source at $x_o = 0$, the distribution of H_z in Region I above the y-axis becomes

$$H_z^I = \sum_{n=0}^{\infty} a_n \cos k_n y\, e^{-jk_x x} . \tag{7}$$

The transverse components of the magnetic and electric field intensities can be derived from H_z by the familiar relations

$$\begin{aligned} \overline{H}_{tr} &= \frac{-jk_z}{\eta^2} \nabla_{tr} H_z, \\ \overline{E}_{tr} &= (\overline{H}_{tr} \times \overline{i}_z)\, \omega\mu/k_z , \end{aligned} \tag{8}$$

where $\nabla_{tr} = \overline{i}_x\, \partial/\partial x = \overline{i}_y\, \partial/\partial y$. Using these equations, the y-component of the electric field intensity becomes

$$E_y^I = \sum_{n=0}^{\infty} \frac{\omega\mu k_x}{\eta^2} a_n \cos k_n y\, e^{-jk_x x} . \tag{9}$$

The corresponding equations in Region II are

$$H_z^{II} = \sum_{n=0}^{\infty} b_n \cos k_n y \, e^{+jk_x x} \quad , \tag{10a}$$

$$E_y^{II} = - \sum_{n=0}^{\infty} \frac{\omega\mu k_x}{\eta^2} b_n \cos k_n y \, e^{+jk_x x} \quad . \tag{10b}$$

Next, the boundary conditions in the yz-plane are applied. These conditions are (Morse and Feshbach)

$$x = 0: \qquad H_z^{I} - H_z^{II} = 0, \tag{11a}$$

$$E_y^{I} - E_y^{II} = - J_m = - I_{zm} \, \delta(y - y_o) \; , \tag{11b}$$

where J_m is the magnetic surface-current density in the z-direction between the Regions I and II at $x = 0$. The first boundary condition, Eq. (11a), yields after substituting of Eqs. (7) and (10a) into (11a) as a relationship between the amplitudes of the waves in Region I and II

$$a_n = b_n \quad .$$

In the second boundary condition, Eq. (11b), a series development is introduced for the delta function,

$$\delta(y - y_o) = \sum_{n=0}^{\infty} \frac{2}{d} \cos k_n y \cos k_n y_o \; ,$$

so that, substituting Eqs. (9) and (10b) into (11b),

$$a_n = - \frac{1}{d} \frac{\eta^2}{\omega\mu k_x} \cos k_n y_o \tag{12}$$

is obtained. The distribution of H_z hence becomes

$$H_z^{I,(II)} = - \sum_{n=0}^{\infty} \frac{1}{d} \frac{\eta^2}{\omega\mu k_x} \cos k_n y_o \cos k_n y \; e^{(\mp) j k_x x} . \qquad (13)$$

The corresponding Green's function at an arbitrary position x_o is then

$$G(x,y \,|\, x_o,y_o) = - \left(\frac{j}{d} \right) \sum_{n=0}^{\infty} \frac{1}{k_o} \cos k_n y_o \cos k_n y \; e^{-jk_x |x-x_o|} , \qquad (14)$$

where

$$k_x = \sqrt{\eta^2 - k_n^2} .$$

TM Waves Excited by a Line Source

The Green's function for this case satisfies the wave equation, Eq. (4). The boundary conditions at the walls are $G(x,0) = G(x,d) = 0$. The corresponding nonhomogeneous wave equation for the longitudinal component of the electric field intensity is

$$\nabla^2 E_z + \eta^2 E_z = j \frac{\eta^2}{\omega\epsilon} I_{ze} \, \delta(x - x_o) \, \delta(y - y_o) , \qquad (15)$$

where I_{ze} is the electric line current which represents a point source in the cross-sectional plane. The Green's function G and E_z are interrelated by

$$G^{TM} = j \frac{\omega\epsilon}{\eta^2} E_z / I_{ze} . \qquad (16)$$

Assuming a source at $x = 0$, the distributions of the electric field intensities in the Regions I and II are

$$E_z^{I(II)} = \sum_{n=1}^{\infty} \begin{Bmatrix} a'_n \\ b'_n \end{Bmatrix} \sin k_n y \; e^{(\mp) j k_x x} . \qquad (17)$$

The transverse components of the fieldstrengths are given by

$$\bar{E}_{tr} = \frac{jk_z}{\eta^2} \nabla_{tr} \bar{E}_z ,$$

and

$$\bar{H}_{tr} = (\bar{i}_z \times \bar{E}_{tr}) \, \omega\epsilon/k_z .$$

These relationships yield for the y-components of the magnetic field intensity

$$H_y^{I(II)} = \sum_{n=1}^{\infty} \left\{ \begin{matrix} a'_n \\ \overline{b'_n} \end{matrix} \right\} \frac{\omega \epsilon k_x}{\eta^2} \sin k_n y \, e^{\mp jk_x x} . \tag{18}$$

The boundary conditions in the yz-plane are

$$x = 0: \qquad E_z^I - E_z^{II} = 0 , \tag{19a}$$

$$H_y^I - H_y^{II} = J_e = I_{ze} \quad \delta (y - y_o) , \tag{19b}$$

where J_e is the density of the electric surface current at the boundary between Region I and II. The first boundary condition gives $a'_n = b'_n$. Substitution of Eq. (18) into (19b), multiplication by $\sin k_n y$, and integration from $y = 0$ to $y = d$ yields after normalization ($I_{ze} = 1$)

$$a'_n = \frac{1}{d} \frac{\eta^2}{\omega \epsilon k_x} \sin k_n y_o .$$

Substitution into Eq. (18) yields for the longitudinal field intensity

$$E_z^{I(II)} = \sum_{n=1}^{\infty} \frac{1}{d} \frac{\eta^2}{\omega \epsilon k_x} \sin k_n y \sin k_n y_o \, e^{\mp jk_x(x-x_o)} . \tag{20}$$

The corresponding Green's function for a point source at x_o, y_o is

$$G(x,y\,|\,x_o,y_o) = \frac{i}{d}\sum_{n=1}^{\infty}\frac{1}{k_x}\sin k_n y \sin k_n y_o\, e^{-ik_n|x-x_o|}. \tag{21}$$

Solution for Arbitrary Excitation

The Eqs. (14) and (21) for the Green's functions can be used for solving the wave equation for arbitrary excitation. The nonhomogeneous wave equation for an arbitrary distribution of the sources has the form

$$\nabla^2 u(x,y) + \eta^2 u(x,y) = g(x_o,y_o). \tag{22}$$

Combination with the corresponding wave equation for the Green's function [Eq. (4)] and simple manipulation yield

$$u(x,y) = \iint G(x,y\,|\,x_o y_o)\,|\,g(x_o y_o)dxdy. \tag{23}$$

Both functions u and G must satisfy the same boundary conditions at $y = 0$ and $y = d$.

Appendix

Source at Arbitrary Position

If the source is located at a position x_o, the Green's functions have changed accordingly. Under these conditions, x becomes a transformed coordinate x', where $x' = x - x_o$. For the Region I, the exponential function in Eq. 13 has then the form $e^{-jk_x(x-x_o)}$. For the Region II, the corresponding wave function is $e^{+jk_x x'} = e^{+jk_x(x-x_o)}$. Since x is smaller than x_o, it can be written

$$e^{+jk_x(x-x_o)} = e^{-jk_x|x-x_o|} \quad .$$

The Green's function valid for both regions can then be expressed in the form as indicated in Eq. 14.

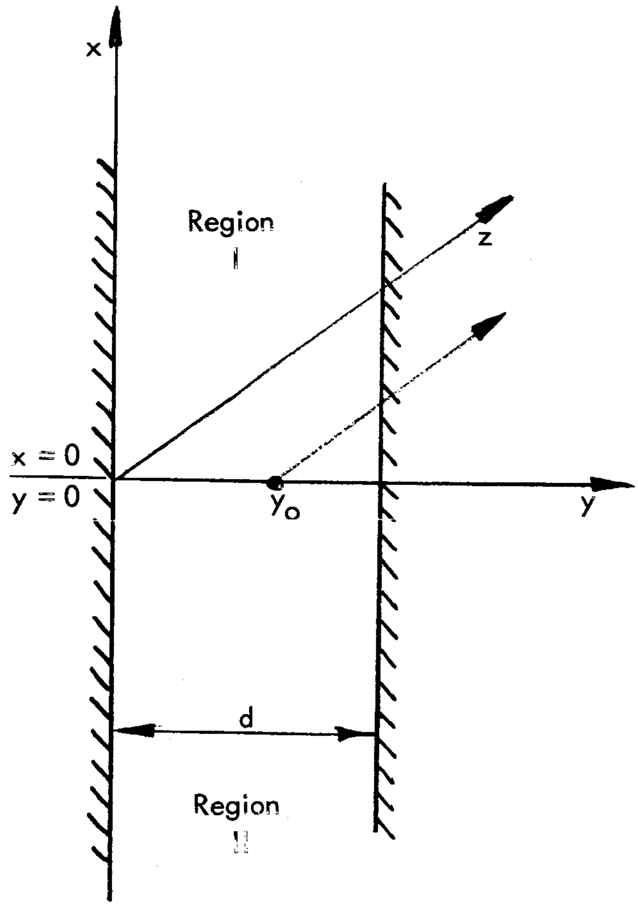

Fig. 1 - Parallel-wall region.